Core 4 - January 2006

① a) i) $f(1) = 3(1)^3 + 2(1)^2 - 7(1) + 2 = 0$

ii) $f(-2) = 3(-2)^3 + 2(-2)^2 - 7(-2) + 2 =$

$= -24 + 8 + 14 + 2 = 0$

iii) $3x^3 + 2x^2 - 7x + 2 = (x+2)(x-1)(3x-1)$

to get $3x^3$ to get $+2$

check: $f(1/3) = 0$ ✓

$$\frac{(x-1)(x+2)}{(x+2)(x-1)(3x-1)} = \frac{1}{3x-1}$$

b) $g(x) = 3x^3 + 2x^2 - 7x + d$

$g(1/3) = 3(1/3)^3 + 2(1/3)^2 - 7(1/3) + d = 2$

$1/9 + 2/9 - 7/3 + d = 2$

$-2 + d = 2 \rightarrow d = 4$

② a) $x = 3 - 4t$ $y = 1 + 2/t = 1 + 2t^{-1}$

$dx/dt = -4$ $dy/dt = -2t^{-2} = -2/t^2$

$dy/dx = dy/dt \times dt/dx$

$= 2/t^2 \times -1/4 = 1/2t^2$

b) when $t = 2$:

$x = 3 - 4(2) = -5$

$y = 1 + 2/2 = 2$

$dy/dx = 1/2(2)^2 = 1/8$

$y - y_1 = m(x - x_1)$

$y - 2 = 1/8(x + 5)$

$8y - 16 = x + 5$

$\rightarrow x - 8y + 21 = 0$

③ $3\cos(\theta) - 2\sin(\theta) = R\cos(\theta + \alpha)$

$= R[\cos\theta\cos\alpha - \sin\theta\sin\alpha]$

$3 = R\cos(\alpha)$
$2 = R\sin(\alpha)$

a) $R = \sqrt{3^2 + 2^2} = \sqrt{13}$ b) $\tan(\alpha) = 2/3 \rightarrow \alpha = 33.690...°$
$\simeq 33.7°$

c) $\sqrt{13}\cos(\theta + 33.7°)$

stretch SF $\sqrt{13}$ in y direction

translation $\begin{pmatrix} -33.7 \\ 0 \end{pmatrix}$

Max value $= \sqrt{13}$

cos has max values at 0, 360, 720 ...

So following translation, max value is at $360 - 33.7$

$= 326.3°$

(4) $V = Ak^t$

a) when $t = 0$, $V = 80 \rightarrow A = 80$

b) when $t = 56$ $V = 5000 \rightarrow 5000 = 80k^{56}$

$5000/80 = k^{56}$

~~$\ln(5000/80) = k$~~

$k = \sqrt[56]{\dfrac{5000}{80}} = 1.076636...$

c) i) $V = 80 \times 1.07663^{106} = £200,647.75$

ii) $800\,000 = 80 \times 1.07663^t$

$\dfrac{800\,000}{80} = 1.07663^t$

$\ln(10\,000) = t\ln(1.07663)$

$\dfrac{\ln(10\,000)}{\ln(1.07663)} = t = 124.7$ years $= 2024$

(5) a) i) $(1-x)^{-1} \approx 1 + (-1)(-x) + \dfrac{(-1)(-2)(-x)^2}{2}$

$\approx 1 + x + x^2$

ii) $\dfrac{1}{3-2x} = (3-2x)^{-1} = 3^{-1}(1 - 2/3(x))^{-1}$

$= 1/3\left[1 + 2/3(x) + (2/3(x))^2\right]$

$= 1/3[1 + 2/3(x) + 4/9(x)^2]$

$= 1/3 + 2/9x + 4/27x^2$

b) $\frac{1}{(1-x)^2} = (1-x)^{-2} \approx 1 + (-2)(-x) + \frac{(-2)(-3)}{2}(-x)$

$\approx 1 + 2x + 3x^2$

c) $\frac{2x^2-3}{(3-2x)(1-x)^2} \equiv \frac{A}{(3-2x)} + \frac{B}{(1-x)} + \frac{C}{(1-x)^2}$

$2x^2 - 3 = A(1-x)^2 + B(3-2x)(1-x) + C(3-2x)$

$\boxed{x=1}$ $\quad -1 = C(1) \rightarrow C = -1$

$\boxed{x=3/2}$ $\quad 3/2 = A(1/4) \rightarrow A = 6$

$\boxed{x=0}$ $\quad -3 = 6(1) + B(3)(1) - 1(3)$

$-3 = 6 + 3B - 3$

$-6 = 3B \rightarrow B = -2$

$\rightarrow \frac{6}{3-2x} - \frac{2}{(1-x)} - \frac{1}{(1-x)^2}$

d) • $6(3-2x)^{-1} = 6/3 + 12/9\, x + 24/27\, x^2$ (FROM a))

• $-2(1-x)^{-1} = -2 - 2x - 2x^2$ (FROM a))

• $-1(1-x)^{-2} = -1 - 2x - 3x^2$ (FROM b))

Add terms together $\rightarrow -1 - 8/3\, x - 37/9\, x^2$

6 a) $\cos(2x) = 2\cos^2(x) - 1$

b) $\int_0^{\pi/2} \cos^2(x)\, dx$

$2\cos^2(x) - 1 = \cos(2x)$

$2\cos^2(x) = \cos(2x) + 1$

$\cos^2(x) = 1/2\,(\cos(2x) + 1)$

$= 1/2 \int_0^{\pi/2} \cos(2x) + 1 \, dx$

$= 1/2 \left[1/2 \sin(2x) + x \right]_0^{\pi/2}$

$= 1/2 \left[1/2 \sin(\pi) + \pi/2 - 1/2 \sin(0) - 0 \right]$

$= 1/2 \left[0 + \pi/2 \right]$

$= \pi/4$

7) a) i) $\vec{AB} = \vec{AO} + \vec{OB}$

$$= \begin{pmatrix} -2 \\ -1 \\ -3 \end{pmatrix} + \begin{pmatrix} 6 \\ 5 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ 0 \end{pmatrix}$$

ii) $\begin{pmatrix} 4 \\ 4 \\ 0 \end{pmatrix} = 4\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ therefore parallel

iii)
$$\left.\begin{aligned} 6 + 1\lambda &= 2 \\ 1 + 1\lambda &= -3 \\ -1 + 0\lambda &= -1 \end{aligned}\right\} \text{all satisfied when } \lambda = -4$$

b) i) $\vec{DM} = \vec{DO} + \vec{OM}$

$$= \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix} + \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix}$$

$$l_2 = \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix}$$

ii) $\vec{AC} = \cancel{\vec{OO}}\ \vec{AO} + \vec{OC}$

$$= \begin{pmatrix} -2 \\ -1 \\ -3 \end{pmatrix} + \begin{pmatrix} 6 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ -4 \end{pmatrix}$$

$\boxed{\text{Scalar}}$ $a.b = \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 0 \\ -4 \end{pmatrix} = 8 + 0 - 8 = 0$

$\therefore$ Angle is $90°$ as vectors are perpendicular

8) a) $dx/dt = -2(x-6)^{1/2}$

$$\frac{1}{(x-6)^{1/2}}\,dx = -2\,dt$$

$$\int (x-6)^{-1/2}\,dx = \int -2\,dt$$

$$2(x-6)^{1/2} = -2t + c$$

$x = 70,\ t = 0 \rightarrow 2\sqrt{64} = c \rightarrow c = 16$

$$2\sqrt{x-6} = -2t + 16$$

$$2t = 16 - 2\sqrt{x-6}$$

$$t = 8 - \sqrt{x-6}$$

b) i) $x = 6 \rightarrow dx/dt = 0 \rightarrow$ fuel stops flowing

ii) $x = 22 \qquad t = 8 - \sqrt{x - 6}$

$= 8 - \sqrt{22 - 6} = 8 - \sqrt{16} = 4$

time $= 4$ minutes